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The Stochastic Boltzmann Equation and Hydrodynamic Fluctuations

Hiroshi Ueyama[1,2]

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Based on the assumption of a kinetic equation in Γ space, a stochastic differential equation of the one-particle distribution is derived without the use of the linear approximation. It is just the Boltzmann equation with a Langevin-fluctuating force term. The result is the general form of the linearized Boltzmann equation with fluctuations found by Bixon and Zwanzig and by Fox and Uhlenbeck. It reduces to the general Landau–Lifshitz equations of fluid dynamics in the presence of fluctuations in a similar hydrodynamic approximation to that used by Chapman and Enskog with respect to the Boltzmann equation.

KEY WORDS: Stochastic differential equation; Langevin equation; Boltzmann equation; hydrodynamic fluctuations; master equation; kinetic equation; hard-sphere system.

1. INTRODUCTION

The general theory of fluctuations in fluid dynamics is constructed by introducing the appropriate additional terms into the general equations of fluid dynamics. Landau and Lifshitz observed that the stress tensor, or the heat current, contains the spontaneous local component that they called the random quantity in addition to the usual component that depends on the velocity, or the temperature, gradient.[1] Properties of the random quantities are determined on the basis of the formulas of fluctuation theory.[2,3]

Since the general equations of fluid dynamics are derivable from the Boltzmann equation, it is expected that in the presence of fluctuations the latter can be extended to a stochastic form. From this point of view, Fox

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[1] Humboldt Fellow. Max-Planck Institut für Festkörperforschung, Stuttgart, West Germany.

[2] Present address: Department of Physics, College of General Education, Osaka University, Toyonaka, Japan.

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and Uhlenbeck[4] and Bixon and Zwanzig[5] presented a theory of the stochastic Boltzmann equation almost at the same time. With the use of the general theory of linear processes due to Onsager and Machlup,[3] they showed that a Langevin-fluctuating force term is to be added to the linearized Boltzmann equation and that this term yields the random quantities of Landau and Lifshitz in the hydrodynamical approximation.

However, questions[6–8] remained with regard to their theory: The physical origin of the fluctuating force is unclear and the stochastic Boltzmann equation does not give the complete form of the equations of Landau and Lifshitz, because of the linear approximation.[4] Attempts have been made at improving the theory[8]: Van Kampen applied the method of the system-size expansion[10] to the problem[9] and Logan and Kac developed a theory[11] based on a coarse-grained master equation. These theories are free from the linear approximation, but are limited to spatially homogeneous fluctuations. It is the open problem of nonlinear and local fluctuations that this paper aims at clarifying.

Further, there is another problem in the theory of Landau and Lifshitz itself: Is it applicable in the nonlinear region? Since the theory is based on the general theory of linear fluctuations,[3] it is formally restricted to the linear region, where the nonlinearity of hydrodynamic equations is negligible. Some authors[8,12,13] feel that it is not restricted. Hinton[14] tried to show that the theory is correct not only near the absolute equilibrium, but also near the local equilibrium. He assumed an equation for the deviation of the distribution from the "local" equilibrium that is similar to, but different from, the linearized Boltzmann equation. There seems to be much room for discussion on that equation.

This paper aims at clarifying the above two problems, i.e., to establish the stochastic Boltzmann equation without the use of the linear approximation and to show the validity of the theory of Landau and Lifshitz in the nonlinear region.

From the viewpoint of the mathematical theory of stochastic processes,[15] the existing theories on the fluctuations of the Boltzmann equation[8,10,11] have a common character: They try to describe the fluctuations in terms of Brownian motion, i.e., as a Gaussian Markov process.[16] First they find a Fokker–Planck equation for the fluctuations of the distribution function and then translate it into a Langevin equation.

There may be another approach: first, to find the set of Langevin-type equations that describes a fluid, or more precisely, a system of particles with short-range interactions, at the "kinetic stage,"[17] and then to construct the Langevin-type equation of the one-particle distribution.

The fluid at the kinetic stage is well described by the master equation[18] or by similar kinetic equations. The master equation describes not a Gaus-

sian, but a Poisson process, as Brout noted.[19,20] The Langevin-type equation (the stochastic differential equation) is popular in the theory of Gaussian processes, but it can also be used to describe Poisson processes.[15,21] Accordingly, it is possible to find a set of Langevin (-type) equations of a system when the master equation of the system is given.

This approach was taken in a previous paper by the author[22] and in a recent one by Onuki.[23] In the former, the Langevin (-type) equation that is stochastically equivalent to the master equation was derived from the microscopic equations of motion, but a slight incorrectness was inevitable in the expressions for correlation functions because of the well-known difficulty[24] in deriving the master equation from the Liouville equation rigorously. In the latter work,[23] the expressions are improved to some extent with the aid of simplifications of the dynamics.

The present paper adopts the master equation as a fundamental assumption. Thus, it is free from the difficulty of deriving the master equation rigorously from the microscopic dynamics. In order to discuss the inhomogeneous fluctuations, another kinetic equation in phase space is assumed as a generalization of the master equation. A set of Langevin (-type) equations (stochastic differential equations) is so constructed that it produces the same expectation value of a physical quantity as the kinetic equation gives. The stochastic Boltzmann equation is derived as an equivalent expression to the Langevin (-type) equations of one-particle quantities.

It is also shown that the stochastic Boltzmann equation derived in this manner reduces to that of Fox and of Bixon[4,5] when it is linearized near equilibrium. The analog to the first approximation of the Chapman–Enskog expansion[25,26] in the stochastic generalization of the Boltzmann equation is shown to be the fundamental equations of fluid dynamics of Landau and Lifshitz. This result implies that their theory is correct in the nonlinear region as long as the collision frequency is so large that the hydrodynamical description of the system is meaningful.

2. FUNDAMENTAL ASSUMPTIONS. KINETIC EQUATIONS IN PHASE SPACE

The homogeneous fluctuations are well described by the master equation[18,19]

$$\frac{\partial}{\partial t} f(\boldsymbol{p}) = \int \{w(\boldsymbol{p}, \boldsymbol{p}') f(\boldsymbol{p}') - w(\boldsymbol{p}', \boldsymbol{p}) f(\boldsymbol{p})\} \, d\boldsymbol{p}' \tag{1}$$

where $\boldsymbol{p}$ is the master vector

$$\boldsymbol{p} = (\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_N), \qquad d\boldsymbol{p} = \prod_{j=1}^{N} d\mathbf{v}_j \tag{2}$$

and $w(p, p')$ takes the form

$$w(p, p') = \frac{\nu}{N} \sum_{(ij)} W(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_i', \mathbf{v}_j') \prod_{k \neq i,j} \delta(\mathbf{v}_k - \mathbf{v}_k') \tag{3}$$

when the two-particle collisions are dominant in a system of N particles. In Eq. (3), ν is a constant. Equation (1) describes a $3N$-dimensional Poisson Markov process, as Brout[19] noticed.

The master equation reduces to the Boltzmann equation (of the homogeneous system) on the assumption of molecular chaos

$$f^{(2)}(\mathbf{v}_1, \mathbf{v}_2, t) \simeq f^{(1)}(\mathbf{v}_1, t) f^{(1)}(\mathbf{v}_2, t) \tag{4}$$

where

$$f^{(s)}(\mathbf{v}_1, \ldots, \mathbf{v}_s, t) \equiv \frac{N!}{(N-s)!} \int \cdots \int f(\mathbf{v}_1, \ldots, \mathbf{v}_N, t) \prod_{k>s} d\mathbf{v}_k \tag{5}$$

Inhomogeneous fluctuations cannot be described by the master equation. The kinetic equation of these fluctuations is expected to have the form

$$\frac{\partial}{\partial t} f(p) + \sum_{j=1}^{N} \mathbf{v}_j \frac{\partial}{\partial \mathbf{x}_j} f(p) = \int \{w(p, p') f(p') - w(p', p) f(p)\} \, dp' \tag{6}$$

where

$$p = (\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, \ldots, \mathbf{x}_N, \mathbf{v}_N)$$

and

$$w(p, p') = \sum W(\mathbf{v}_i, \mathbf{v}_j, \mathbf{v}_i', \mathbf{v}_j'; \mathbf{x}_i, \mathbf{x}_j) \, \delta(\mathbf{x}_i - \mathbf{x}_i') \, \delta(\mathbf{x}_j - \mathbf{x}_j') \times \prod_{k \neq i,j} \delta(\mathbf{x}_k - \mathbf{x}_k') \, \delta(\mathbf{v}_k - \mathbf{v}_k') \tag{7}$$

There is no factor of $1/N$ in Eq. (7), in contrast to Eq. (3), because the factor comes from the spatial distribution $1/V$, where V is the volume of the system. It is expected that Eq. (6) yields the Boltzmann equation on the assumption of molecular chaos. This condition specifies the form of W as

$$W(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1', \mathbf{v}_2'; \mathbf{x}_1, \mathbf{x}_2') = W_B(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1', \mathbf{v}_2') \, \delta(\mathbf{x}_1 - \mathbf{x}_2) \tag{8}$$

where

$$W_B(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1', \mathbf{v}_2') = g\sigma(g, \theta) \, \delta(\mathbf{v}_1 + \mathbf{v}_2 - \mathbf{v}_1' - \mathbf{v}_2') \, \delta\left(\frac{g^2 - g'^2}{2}\right) \tag{9}$$

with the use of the collisional cross section σ and $\mathbf{g} = \mathbf{v}_1 - \mathbf{v}_2$.

There is a conceptual difficulty in the generalization of the master equation to Eq. (6): Since the positions and the velocities of all the particles are specified by $\boldsymbol{p}$, there is no room to interpret $f(\boldsymbol{p})$ in terms of probability.[18] In spite of this sophisticated problem, Eq. (6) is known to be a good approximation of the Liouville equation, at least for the hard-sphere system. Hopfield and Bastin[27] have intuitively introduced Eq. (6) and Ernst *et al.*[28] have derived it from the Liouville equation.

They give Eq. (6) with an expression for W different from Eq. (8),

$$W(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_1', \mathbf{v}_2'; \mathbf{x}_1, \mathbf{x}_2)$$
$$= \sigma^2 \int_{\mathbf{v}_{21}\cdot\boldsymbol{\sigma}>0} d\boldsymbol{\sigma}\,(\mathbf{v}_{21}, \boldsymbol{\sigma})\{\delta(\mathbf{x}_1' - \mathbf{x}_2' - \boldsymbol{\sigma})\,\delta(\mathbf{v}_1 - \mathbf{v}_1^*)\,\delta(\mathbf{v}_2 - \mathbf{v}_2^*)$$
$$- \delta(\mathbf{x}_1' - \mathbf{x}_2' + \boldsymbol{\sigma})\,\delta(\mathbf{v}_1 - \mathbf{v}_1')\,\delta(\mathbf{v}_2 - \mathbf{v}_2')\} \tag{10}$$

where

$$\mathbf{v}_1^* = \mathbf{v}_1' - (\mathbf{v}_{12}'\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}, \qquad \mathbf{v}_2^* = \mathbf{v}_2' + (\mathbf{v}_{12}'\cdot\boldsymbol{\sigma})\boldsymbol{\sigma}$$

Equation (10) reduces to Eq. (8) for dilute systems. For systems with different types of interactions, Eq. (6) is expected to still be a good approximation of the Liouville equation, as long as the collision duration is sufficiently small compared to the mean free time.

Equations (1) and (6) are adopted in this paper as the fundamental assumptions for the discussion of homogeneous and inhomogeneous fluctuations, respectively.

3. STOCHASTIC DIFFERENTIAL EQUATION ASSOCIATED WITH THE MASTER EQUATION

Now we construct the set of Langevin (-type) equations that is stochastically equivalent to the kinetic equation. The stochastic Boltzmann equation is derived from these Langevin equations in the following sections.

The term Langevin equation is often used in a mathematically well defined, restrictive sense, but the original equation that Langevin introduced[29] is nothing but the Newtonian equation of motion of a particle immersed in a fluid. We try to find the equation of motion of each particle of a system, given the kinetic equation of the system.

The master equation is a Poisson process[19] and mathematicians[15,21,30] have given the stochastic differential equation of the process. But we keep to the view that the Langevin or the stochastic differential equation is nothing but the equation of motion.

If Eq. (1) is solved, the distribution function $f(\mathbf{p}, t)$ gives the expectation value of the arbitrary quantity $F = F(\boldsymbol{p})$ as follows:

$$\langle F \rangle_t = \int F(\boldsymbol{p}) f(\boldsymbol{p}, t)\, d\boldsymbol{p} \tag{11}$$

When the function at a time t is specified as

$$f(\boldsymbol{p}, t) = \delta(\boldsymbol{p} - \boldsymbol{v}_t) \tag{12}$$

with the aid of a parameter $\boldsymbol{v}_t$, then the expectation value at a later time $t + \tau$ after a small interval τ is given by

$$\langle F \rangle_{t+\tau} = F(\boldsymbol{v}_t) + \tau \int \{F(\boldsymbol{p}) - F(\boldsymbol{v}_t)\} w(\boldsymbol{p}, \boldsymbol{v}_t)\, d\boldsymbol{p} + O(\tau^2) \tag{13}$$

For the moment, we restrict ourselves to the case in which $\boldsymbol{p}$ is one-dimensional: $\boldsymbol{p} = p$ and $\boldsymbol{v}_t = v_t$. Equation (13) gives

$$\langle p \rangle_{t+\tau} = v_t + \tau \alpha_1(v_t) + O(\tau^2) \tag{14}$$

as a special case, where the derivate moments[31] are defined by

$$\alpha_n(p) = \int (p' - p)^n w(p', p)\, dp'; \qquad n = 1, 2, 3,... \tag{15}$$

Equation (14) describes the change of velocity in a small interval similarly to the Newtonian equation of motion, but it fails to give the expectation values of other physical quantities.

Intuitively, we assume that the equation of motion is given by

$$v_{t+\tau} = v_t + \tau \alpha_1(v_t) + \xi(v_t, \tau) \tag{16}$$

with a random quantity $\xi = \xi(v_t, \tau)$ and ask for the condition by which the average of an arbitrary quantity over this random quantity is equal to the expectation value of Eq. (13):

$$\langle F \rangle_{t+\tau} = \overline{F(v_{t+\tau})} \tag{17}$$

As a special case, Eq. (17) gives

$$\overline{\xi(v_t, \tau)} = 0 \tag{18}$$

with the use of Eqs. (14) and (16). Without loss of generality, we may assume that the function F is analytic and that Eq. (17) is required only for an arbitrary monomial. This requirement is satisfied if the following conditions hold:

$$\begin{aligned} \overline{\xi^k(v_t, \tau)} &= 0 \qquad \text{if} \quad k = 1 \\ &= \tau \alpha_k(v_t) \qquad \text{if} \quad k = 2, 3,... \end{aligned} \tag{19}$$

We may conclude that Eq. (16) together with (19) is stochastically equivalent to Eq. (13) in the sense of Eq. (17) on the condition that Eq. (12) is satisfied.

We may continue this method of constructing Eq. (16) for successive time steps, since the distribution function at $t + \tau$ is written as

$$f(p, t + \tau) = \overline{\delta(p - v_{t+\tau})} \tag{20}$$

We may construct Eqs. (16) and (19) in which t is replaced by $t + \tau$, considering $f(p, t + \tau)$ as a δ-function, and finally take the average over $\xi(v_t, \tau)$. It is clear that the random quantities of different time steps are mutually independent.

Now, we may conclude that Eq. (16) is stochastically equivalent to Eq. (1) as far as the time discretization approximation[16] is concerned. We may take the limit $\tau \to 0$ naively and from Eq. (16) obtain the stochastic differential equation

$$\frac{d}{dt} v_t = \alpha_1(v_t) + \xi'(v_t, t) \tag{21}$$

where $\xi'(t) = \xi'(v_t, t)$ is defined through the relation

$$\int_t^{t+\tau} \xi'(v_s, s)\, ds = \xi(v_t, \tau) \tag{22}$$

Since the $\xi(v_t, \tau)$ of different time steps are mutually independent, $\xi'(t)$ is a δ-correlated process,

$$\begin{aligned} \overline{\xi'(t)} &= 0 \\ \overline{\xi'(t)\xi'(t_1) \cdots \xi'(t_n)} &= \delta(t - t_1)\, \delta(t_1 - t_2) \cdots \delta(t_{n-1} - t_n)\alpha_{n+1}(v_t) \end{aligned} \tag{23}$$

Equations (21) and (23) are the desired results.

It is interesting to consider the time development of the "distribution" defined by

$$\boldsymbol{g}(\boldsymbol{p}, t) = \delta(\boldsymbol{p} - \boldsymbol{v}_t) \tag{24}$$

It is related to the "distribution function" through Eq. (20). With the use of the time discretization approximation of Eq. (16), we may construct the following Langevin-type equation for $\boldsymbol{g}(\boldsymbol{p}, t)$:

$$\frac{\partial}{\partial t} \boldsymbol{g}(\boldsymbol{p}, t) = \int \{\boldsymbol{w}(\boldsymbol{p}, \boldsymbol{p}')\boldsymbol{g}(\boldsymbol{p}', t) - \boldsymbol{w}(\boldsymbol{p}', \boldsymbol{p})\boldsymbol{g}(\boldsymbol{p}, t)\}\, d\boldsymbol{p} + \boldsymbol{r}(\boldsymbol{p}, t) \tag{25}$$

where the new random quantity is characterized by

$$\begin{aligned}
\overline{\boldsymbol{r}(\boldsymbol{p}, t)} &= 0 \\
\overline{\boldsymbol{r}(\boldsymbol{p}_1, t_1)\boldsymbol{r}(\boldsymbol{p}_2, t_2)} &= \delta(t_1 - t_2)\iint d\boldsymbol{p}\, d\boldsymbol{p}' \\
&\quad \times \left\{\prod_{i=1}^{2}[\delta(\boldsymbol{p}_i - \boldsymbol{p}) - \delta(\boldsymbol{p}_i - \boldsymbol{p}')]\right\} \\
&\quad \times \boldsymbol{w}(\boldsymbol{p}, \boldsymbol{p}')\boldsymbol{g}(\boldsymbol{p}', t_1) \qquad (26) \\
\overline{\boldsymbol{r}(\boldsymbol{p}_1, t_1)\boldsymbol{r}(\boldsymbol{p}_2, t_2)\boldsymbol{r}(\boldsymbol{p}_3, t_3)} &= \delta(t_1 - t_2)\,\delta(t_2 - t_3)\iint d\boldsymbol{p}\, d\boldsymbol{p}' \\
&\quad \times \left\{\prod_{i=1}^{3}[\delta(\boldsymbol{p}_i - \boldsymbol{p}) - \delta(\boldsymbol{p}_i - \boldsymbol{p}')]\right\} \\
&\quad \times \boldsymbol{w}(\boldsymbol{p}, \boldsymbol{p}')\boldsymbol{g}(\boldsymbol{p}', t_1) \\
&\quad \cdots
\end{aligned}$$

Equations (25) and (26) are derived in Appendix B.

Equation (25) has an apparent similarity to Eq. (1): The random quantity $\boldsymbol{r}(\boldsymbol{p}, t)$ is simply added to the right-hand side, which is redundant for one-dimensional processes. But it is important for these many-dimensional processes in which only a limited number of variables are of interest.

It is easy to generalize the above results to many-dimensional processes and prove Eq. (25) and (26) for them:

Equation (16), or Eq. (19), is to be replaced by

$$v_{j\mu t+\tau} = v_{j\mu t} + \tau\alpha_{1j\mu}(\boldsymbol{v}_t) + \xi_{j\mu}(\boldsymbol{v}_t, \tau) \qquad (16')$$

where

$$\boldsymbol{v}_t = (\boldsymbol{v}_{1t}, \boldsymbol{v}_{2t}, \ldots, \boldsymbol{v}_{Nt}), \qquad \alpha_{1j\mu}(\boldsymbol{p}) = \int (p'_{j\mu} - p_{j\mu})\boldsymbol{w}(\boldsymbol{p}', \boldsymbol{p})\, d\boldsymbol{p}'$$

or

$$\begin{aligned}
\overline{\prod_{j\mu}\xi_{j\mu}^{n_{j\mu}}} &= 0 \qquad \text{if} \quad \sum_{j\mu} n_{j\mu} = 1 \\
&= \tau\int\left\{\prod_{j\mu}(p_{j\mu} - v_{j\mu t})^{n_{j\mu}}\right\}\boldsymbol{w}(\boldsymbol{p}, \boldsymbol{v}_t)\, d\boldsymbol{p} \qquad \text{if} \quad \sum_{j\mu} n_{j\mu} > 1
\end{aligned} \qquad (19')$$

Equations (25) and (26) need no modifications only if the delta functions are interpreted as

$$\delta(\boldsymbol{p} - \boldsymbol{v}_t) = \prod_{j\mu}\delta(p_{j\mu} - v_{j\mu t}), \quad \text{etc.} \qquad (27)$$

4. STOCHASTIC BOLTZMANN EQUATION. HOMOGENEOUS CASE

The homogeneous fluctuations are described by Eqs. (1)–(3). In the preceding section, the Langevin-type equation was derived for the distribution. In this section the stochastic Boltzmann equation is derived from that equation.

With the aid of the factorization property of the distribution of Eq. (27), Eq. (25) changes to

$$\frac{\partial}{\partial t}\delta(\boldsymbol{p}_i - \boldsymbol{v}_{it}) = \frac{\nu}{N}\sum_{j\neq i}\iint\{W(\boldsymbol{p}_i, \boldsymbol{p}_j, \boldsymbol{p}_i', \boldsymbol{p}_j')\,\delta(\boldsymbol{p}_i' - \boldsymbol{v}_{it})\,\delta(\boldsymbol{p}_j' - \boldsymbol{v}_{jt}) - W(\boldsymbol{p}_i', \boldsymbol{p}_j', \boldsymbol{p}_i, \boldsymbol{p}_j)\,\delta(\boldsymbol{p}_i - \boldsymbol{v}_{it})\,\delta(\boldsymbol{p}_j - \boldsymbol{v}_{jt})\}\,d\boldsymbol{p}_i'\,d\boldsymbol{p}_j' + r_i(\boldsymbol{p}_i, t) \tag{28}$$

where integrations over $N-1$ variables are carried out. The reduced random quantity introduced in Eq. (28) is

$$r_i(\boldsymbol{p}_i, t) = \int\cdots\int \boldsymbol{r}(\boldsymbol{p}, t)\prod_{k\neq i}\,d\boldsymbol{p}_k \tag{29}$$

The one-particle distribution may be introduced by

$$g(\boldsymbol{p}, t) = \frac{1}{N}\sum_{i=1}^{N}\delta(\boldsymbol{p}_i - \boldsymbol{v}_{it}) \tag{30}$$

where the factor of $1/N$ corresponds to the spatial part of the distribution. Since $W(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_1', \boldsymbol{p}_2')$ vanishes when $\boldsymbol{p}_1' = \boldsymbol{p}_2'$, from Eq. (28) we obtain the stochastic Boltzmann equation

$$\frac{\partial}{\partial t}g(\boldsymbol{p}, t) = \iint\{W(\boldsymbol{p}, \boldsymbol{p}_1, \boldsymbol{p}', \boldsymbol{p}_1')g(\boldsymbol{p}', t)g(\boldsymbol{p}_1', t) - W(\boldsymbol{p}', \boldsymbol{p}_1', \boldsymbol{p}, \boldsymbol{p}_1)g(\boldsymbol{p}, t)g(\boldsymbol{p}_1, t)\} + r(\boldsymbol{p}, t) \tag{31}$$

where

$$r(\boldsymbol{p}, t) = \frac{1}{N}\sum_i r_i(\boldsymbol{p}, t) = \frac{1}{N}\sum_i\int\cdots\int\delta(\boldsymbol{p} - \boldsymbol{p}_i)\boldsymbol{r}(\boldsymbol{p}, t)\prod_{k=1}^{N}\,d\boldsymbol{p}_k \tag{32}$$

The random quantity $r(\boldsymbol{p}, t)$ is characterized through Eqs. (26) and (30).

By carrying out the integrations, we obtain

$$\overline{r(\boldsymbol{p}, t)r(\boldsymbol{p}', t')} = \tfrac{1}{2}\,\delta(t - t')\iint \Delta[\delta_{\mathrm{p}}]\,\Delta[\delta_{\mathrm{p}'}]\,W(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_1', \boldsymbol{p}_2') \times g(\boldsymbol{p}_1', t)g(\boldsymbol{p}_2', t)\,d\boldsymbol{p}_1\,d\boldsymbol{p}_2\,d\boldsymbol{p}_1'\,d\boldsymbol{p}_2' \tag{33}$$

$$\overline{r(\boldsymbol{p}, t)r(\boldsymbol{p}', t')r(\boldsymbol{p}'', t'')} = (1/2N)\,\delta(t - t')\,\delta(t' - t'') \times \iint \Delta[\delta_{\mathrm{p}}]\,\Delta[\delta_{\mathrm{p}'}]\,\Delta[\delta_{\mathrm{p}''}]\,W(\boldsymbol{p}_1, \boldsymbol{p}_2, \boldsymbol{p}_1', \boldsymbol{p}_2')g(\boldsymbol{p}_1', t)g(\boldsymbol{p}_2', t) \times d\boldsymbol{p}_1\,d\boldsymbol{p}_2\,d\boldsymbol{p}_1'\,d\boldsymbol{p}_2' \tag{34}$$

where

$$\Delta[\delta_{\mathrm{p}}] = \delta(\boldsymbol{p} - \boldsymbol{p}_1) + \delta(\boldsymbol{p} - \boldsymbol{p}_2) - \delta(\boldsymbol{p} - \boldsymbol{p}_1') - \delta(\boldsymbol{p} - \boldsymbol{p}_2') \tag{35}$$

In the thermodynamic limit $N \to 0$, the right-hand side of Eq. (34) vanishes. The kth-order correlation function ($k \geqslant 2$) is expected to vanish also, since it is believed to have a factor of N^{2-k}. The random quantity $r(\boldsymbol{p}, t)$ is the Gaussian white noise, which is solely determined by Eq. (33) in this limit.

5. STOCHASTIC BOLTZMANN EQUATION. INHOMOGENEOUS CASE

The kinetic equation is given by Eq. (6). In the local collision approximation of Eq. (8), the position vectors of particles may be considered as parameters of the kinetic equation, at least for a small interval of the order of the collision duration. In addition, we may use the results of Eqs. (25) and (26) also for the inhomogeneous case.

In fact, the Langevin (-type) equation of motion of each particle is given by

$$\begin{aligned} x_{j\mu t+\tau} &= x_{j\mu t} + \tau v_{j\mu t} \\ v_{j\mu t+\tau} &= v_{j\mu t} + \tau\alpha_{1j\mu}(\mathbf{v}_t, \mathbf{x}_t) + \xi_{j\mu}(\mathbf{v}_t, \mathbf{x}_t, t) \end{aligned} \tag{36}$$

in place of Eq. (16′). There is no random force term in the first of Eqs. (36). A slightly different method of deriving the stochastic Boltzmann equation is given in Appendix A for the hard sphere system.

Applying the result of Eq. (25) to Eqs. (1)–(6), we obtain for the distribution in phase space

$$g(\boldsymbol{p}, t) = \prod_{j=1}^{N} \delta(\mathbf{p}_j - \mathbf{v}_{jt})\,\delta(\mathbf{q}_j - \mathbf{x}_{jt})$$

the following stochastic equation:

$$\frac{\partial}{\partial t} g(p, t) + \sum_{j=1}^{N} \mathbf{p}_j \frac{\partial}{\partial \mathbf{q}_j} g(p, t)$$
$$= \int \{w(p, p')g(p', t) - w(p', p)g(p, t)\}\, dp + r(p, t) \tag{37}$$

where $p = (p_1, p_2, ..., p_N, \mathbf{q}_1, \mathbf{q}_2, ..., \mathbf{q}_N)$. Equation (37) may be reduced to an equation of the one-particle quantity.

The one-particle distribution may be defined by

$$g(\mathbf{p}, \mathbf{q}, t) = \sum_{j=1}^{N} \delta(\mathbf{p} - \mathbf{v}_{jt})\, \delta(\mathbf{q} - \mathbf{x}_{jt}) \tag{38}$$

which satisfies the following stochastic Boltzmann equation:

$$\frac{\partial}{\partial t} g(\mathbf{p}, \mathbf{q}, t) + \mathbf{p} \frac{\partial}{\partial \mathbf{q}} g = J(\mathbf{g}) + r(\mathbf{p}, \mathbf{q}, t) \tag{39}$$

where $J(g)$ is the Boltzmann collision operator

$$J(g) = \tfrac{1}{2} \int\int\int \{W_B(\mathbf{p}, \mathbf{p}_1, \mathbf{p}', \mathbf{p}_1')g(\mathbf{p}', \mathbf{q}, t)g(\mathbf{p}_1', \mathbf{q}, t)$$
$$- W_B(\mathbf{p}', \mathbf{p}_1', \mathbf{p}, \mathbf{p}_1)g(\mathbf{p}, \mathbf{q}, t)g(\mathbf{p}_1, \mathbf{q}, t)\}\, d\mathbf{p}_1\, d\mathbf{p}'\, d\mathbf{p}_1' \tag{40}$$

Characterizations of the random quantity $r(\mathbf{p}, \mathbf{q}, t)$ are found from Eq. (26) as follows:

$$\overline{r(\mathbf{p}, \mathbf{q}, t)r(\mathbf{p}', \mathbf{q}', t')}$$
$$= \tfrac{1}{2}\, \delta(t - t')\, \delta(\mathbf{q} - \mathbf{q}') \int \cdots \int d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{p}_1'\, d\mathbf{p}_2'$$
$$\times \Delta[\delta_{\mathbf{p}}]\, \Delta[\delta_{\mathbf{p}'}]\, W_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1', \mathbf{p}_2')g(\mathbf{p}_1', \mathbf{q}, t)g(\mathbf{p}_2', \mathbf{q}, t) \tag{41}$$

$$\overline{r(\mathbf{p}, \mathbf{q}, t)r(\mathbf{p}', \mathbf{q}', t')r(\mathbf{p}'', \mathbf{q}'', t'')}$$
$$= \tfrac{1}{2}\, \delta(t - t')\, \delta(t' - t'')\, \delta(\mathbf{q} - \mathbf{q}')\, \delta(\mathbf{q}' - \mathbf{q}'') \int \cdots \int d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{p}_1'\, d\mathbf{p}_2'$$
$$\times \Delta[\delta_{\mathbf{p}}]\, \Delta[\delta_{\mathbf{p}'}]\, \Delta[\delta_{\mathbf{p}''}]\, W_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1', \mathbf{p}_2')g(\mathbf{p}_1', \mathbf{q}, t)g(\mathbf{p}_2', \mathbf{q}, t) \tag{42}$$

The higher order correlation functions may be taken as

$$\overline{\prod_{\alpha=1}^{n} r(\mathbf{p}_\alpha, \mathbf{q}_\alpha, t_\alpha)} = \tfrac{1}{2}\left\{\prod_{\alpha=1}^{n-1} \delta(t_\alpha - t_{\alpha+1})\, \delta(\mathbf{q}_\alpha - \mathbf{q}_{\alpha+1})\right\}$$
$$\times \int \cdots \int \left\{\prod_{\beta=1}^{n} \Delta[\delta_{\mathbf{p}_\beta}]\right\} W_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1', \mathbf{p}_2')$$
$$\times g(\mathbf{p}_1', \mathbf{q}, t)g(\mathbf{p}_2', \mathbf{q}, t)\, d\mathbf{p}_1\, d\mathbf{p}_2\, d\mathbf{p}_1'\, d\mathbf{p}_2' \tag{43}$$

Compared to the case of homogeneous fluctuations, no factor of $1/N$ appears either in the stochastic Boltzmann equation or in the correlation functions of the random quantity. This fact means that local fluctuations are independent of the size of the whole system and that the random quantity $r(\mathbf{p}, \mathbf{q}, t)$ is not a Gaussian, but a Poissonian-like process.

6. HYDRODYNAMICAL APPROXIMATION

The stochastic Boltzmann equation derived in the preceding section is the same as the Boltzmann equation except for the random quantity. Then, almost all of the theorems on the Boltzmann equation may be proven also for the stochastic analog.

The collision operator has five invariants

$$\psi_0 = 1, \qquad \psi_i = v_i \quad (i = 1, 2, 3), \qquad \psi_4 = \tfrac{1}{2}mv^2$$

from which we may construct the stochastic analogs of the five conserved quantities

$$\rho_\alpha(\mathbf{q}) = \int \psi_\alpha(\mathbf{p}) g(\mathbf{p}, \mathbf{q}, t)\, d\mathbf{p} = \sum_{j=1}^{N} \psi_\alpha(\mathbf{v}_{jt})\, \delta(\mathbf{q} - \mathbf{x}_{jt}) \tag{44}$$

Their time development is given by

$$\frac{\partial}{\partial t} \rho_\alpha(\mathbf{q}) + \frac{\partial}{\partial \mathbf{q}} j_\alpha(\mathbf{q}) = R_\alpha(\mathbf{q}) = 0 \tag{45}$$

where

$$j_\alpha(\mathbf{q}) = \int \mathbf{p} \psi_\alpha(\mathbf{p}) g(\mathbf{p}, \mathbf{q}, t)\, d\mathbf{p} \tag{46}$$

$$R_\alpha(\mathbf{q}) = \int \psi_\alpha(\mathbf{p}) r(\mathbf{p}, \mathbf{q}, t)\, d\mathbf{p} \tag{47}$$

Apparently, the random quantity violates the conservation laws in (45), but it may be considered identically zero because all of the correlation functions of $R_\alpha(\mathbf{q})$ vanish due to the identity

$$\Delta[\psi_\alpha(\mathbf{p})]\, W_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1', \mathbf{p}_2') = 0 \tag{48}$$

We have proven the second identity of (45).

When the collision term is dominant, the distribution is a functional of the conserved quantities, irrespective of the initial condition. The zeroth-order approximate solution of Eq. (39) is the local equilibrium distribution

$$F(\mathbf{q}, \mathbf{C}, t) = n(\beta/2\pi)^{3/2} \exp[-\beta(\mathbf{u} - \mathbf{p})^2] \tag{49}$$

where

$$n = \rho_0, \qquad u_i = \rho_i \quad (i = 1, 2, 3), \qquad \beta = \frac{m}{2kT} = \frac{3m}{4\rho_4}, \qquad \mathbf{C} = \mathbf{p} - \mathbf{u}$$

The local equilibrium distribution defined by Eq. (49) is formally equivalent to the usual one, but there is a slight difference because of the different definitions of the conserved quantities. This difference is important when the hard-sphere system is considered in Appendix A.

Following the spirit of the Chapman–Enskog expansion, the first-order approximation is given by

$$g(\mathbf{p}, \mathbf{q}, t) = F(\mathbf{q}, \mathbf{C}, t)[1 + \Phi(\mathbf{q}, \mathbf{C}, t)] \tag{50}$$

where Φ is a "small" deviation. When Eq. (50) is substituted and higher order terms in Φ are neglected, Eq. (39) changes to

$$\mathscr{D}[F] = -n^2\mathscr{J}[\Phi] + r(\mathbf{p}, \mathbf{q}, t) \tag{51}$$

where

$$\mathscr{D}[F] = \frac{\partial}{\partial t}F + \mathbf{p}\frac{\partial}{\partial \mathbf{q}}F \tag{52}$$

$$n^2\mathscr{J}[\Phi] = -\int \cdots \int d\mathbf{p}_1\, d\mathbf{p}'\, d\mathbf{p}_1'\, W(\mathbf{p}, \mathbf{p}_1, \mathbf{p}', \mathbf{p}_1')F(\mathbf{q}, \mathbf{C}, t)F(\mathbf{q}, \mathbf{C}', t)$$
$$\times \{\Phi(\mathbf{C}') + \Phi(\mathbf{C}_1') - \Phi(\mathbf{C}) - \Phi(\mathbf{C}_1)\} \tag{53}$$

Equation (50) is also to be substituted into Eqs. (41)–(43). Since the zeroth-order terms survive, higher order terms in Φ may be neglected and the following substitution is sufficient for Eqs. (41)–(43):

$$g(\mathbf{p}_1', \mathbf{q}, t)g(\mathbf{p}_2', \mathbf{q}, t) \simeq F(\mathbf{q}, \mathbf{C}_1, t)F(\mathbf{q}, \mathbf{C}_2, t) \tag{54}$$

Since the collision operator $\mathscr{J}$ is a linear integral operator, Eq. (51) may be solved in the form

$$\Phi = \Phi_1 + \Phi_2$$

where

$$n^2\mathscr{J}[\Phi_1] = -\mathscr{D}[F] \tag{55}$$

$$n^2\mathscr{J}[\Phi_2] = r(\mathbf{p}, \mathbf{q}, t) \tag{56}$$

The stress tensor $p_{\mu\nu}$ or the heat current q_μ is a linear functional of

$$p_{\mu\nu} - p\delta_{\mu\nu} = \int \overline{C_\mu C_\nu} F\Phi\, d\mathbf{C} \tag{57}$$

or

$$q_\mu = (m/2\beta)\int (\beta C^2 - \tfrac{5}{2})C_\mu F\Phi \, d\mathbf{C} \tag{58}$$

and accordingly is composed of the two Φ contributions as follows:

$$p_{\mu\nu} - p\delta_{\mu\nu} = p_{1\mu\nu} + p_{2\mu\nu} \qquad \text{or} \qquad q_\mu = q_{1\mu} + q_{2\mu} \tag{59}$$

In Eq. (57), the following tensor notation[26] is used

$$\overline{C_\mu C_\nu} = C_\mu C_\nu - \tfrac{1}{3}C^2$$

The first Φ component of $p_{\mu\nu}$ and q_μ is given by the traditional expressions. Following the notations of Waldmann,[26] they are given as follows:

$$p_{1\mu\nu} = -\eta\left(\frac{\partial v_\mu}{\partial x_\nu} + \frac{\partial v_\nu}{\partial x_\mu} - \frac{2}{3}\delta_{\mu\nu}\frac{\partial v_\lambda}{\partial x_\lambda}\right), \qquad q_{1\mu} = -\lambda\frac{\partial T}{\partial x_\mu} \tag{60}$$

where the transport coefficients are given by

$$\eta = \tfrac{1}{10}kT\beta^2\int \mathscr{J}[B\overline{C_\mu C_\nu}]B\overline{C_\mu C_\nu}\, d\mathbf{C}, \qquad \lambda = \tfrac{1}{3}k\int \mathscr{J}[AC_\mu]AC_\mu \, d\mathbf{C} \tag{61}$$

with the use of the solutions $A = A(C)$ and $B = B(C)$ of the integral equations

$$\mathscr{J}[AC_\mu] = (F/n)(\beta C^2 - \tfrac{5}{2})C_\mu, \qquad \mathscr{J}[B\overline{C_\mu C_\nu}] = (2F/n)\overline{C_\mu C_\nu} \tag{62}$$

The second Φ component of $p_{\mu\nu}$ and q_μ represents the fluctuations. Substituting the formal solution of Eq. (56) into Eq. (59), we obtain the following expressions for the fluctuating components in terms of the random quantity of the stochastic Boltzmann equation:

$$\begin{aligned} p_{2\mu\nu} &= m\int \overline{C_\mu C_\nu}(F/n^2)\mathscr{J}^{-1}[r(\mathbf{p}, \mathbf{q}, t)]\, d\mathbf{C} \\ &= (m/2n)\int \mathscr{J}^{-1}[(F/n^2)\overline{C_\mu C_\nu}]r(\mathbf{p}, \mathbf{q}, t)\, d\mathbf{C} \\ &= (m/2n)\int B\overline{C_\mu C_\nu}r(\mathbf{p}, \mathbf{q}, t)\, d\mathbf{C} \end{aligned} \tag{63}$$

$$\begin{aligned} q_{2\mu} &= (m/2\beta)\int (\beta C^2 - \tfrac{5}{2})C_\mu(F/n^2)\mathscr{J}^{-1}[r(\mathbf{p}, \mathbf{q}, t)]\, d\mathbf{C} \\ &= (m/2n\beta)\int AC_\mu r(\mathbf{p}, \mathbf{q}, t)\, d\mathbf{C} \end{aligned} \tag{64}$$

Equations (63) and (64) represent the relation between the fluctuations of the fluid dynamics and those of the Boltzmann equation, just as Eq. (57)

represents the relation between the hydrodynamic quantities and the distribution function.

With the aid of Eqs. (41) and (54), Eqs. (63) and (64) yield the following relations:

$$\begin{aligned}
\overline{p_{2\mu\nu}(\mathbf{q}, t)} &= 0, \qquad \overline{q_{2\mu}(\mathbf{q}, t)} = 0 \\
\overline{p_{2\mu\nu}(\mathbf{q}, t)p_{2\mu'\nu'}(\mathbf{q}', t')} &= 2\eta kT(\delta_{\mu\mu'}\delta_{\nu\nu'} + \delta_{\mu\nu'}\delta_{\nu\mu'} - \tfrac{3}{2}\delta_{\mu\nu}\delta_{\mu'\nu'}) \\
&\quad \times \delta(t - t')\,\delta(\mathbf{q} - \mathbf{q}') \qquad (65) \\
\overline{q_{2\mu}(\mathbf{q}, t)q_{2\mu'}(\mathbf{q}', t')} &= 2\lambda kT^2\,\delta_{\mu\mu'}\,\delta(t - t')\,\delta(\mathbf{q} - \mathbf{q}') \\
\overline{q_{2\mu}(\mathbf{q}, t)p_{2\mu'\nu'}(\mathbf{q}', t')} &= 0
\end{aligned}$$

These relations are exactly the same as those of Landau and Lifshitz.[1]

The random quantity $r(\mathbf{p}, \mathbf{q}, t)$, however, is not a Gaussian, but a Poissonian-like process. If Eq. (63) or Eq. (64) is formally substituted into Eq. (43), there arise many complicated expressions of higher order correlation functions of hydrodynamic quantities. The fundamental point of the hydrodynamic approximation lies in the fact that collisions occur so frequently that the deviation from the local equilibrium is small. Near the local equilibrium, all the physical variables of concern are approximately functionals of the conserved quantities that satisfy Eq. (48). Accordingly, not only is the substitution of Eq. (54) permitted, but in addition the factor $\Delta[\delta_p]$ in Eq. (43) may be considered a small quantity. The third and higher order correlation functions of the pressure tensor and of the heat current are to be neglected, and $p_{2\mu\nu}$ and $q_{2\mu}$ are Gaussian processes.

The equations of Landau and Lifshitz are derived on a different basis. These equations are valid not only near the absolute equilibrium, but also near the local equilibrium.

The above method of deriving the equations of Landau and Lifshitz is different from that of Hinton.[14] Equation (51) may be compared to his linearized Boltzmann equation, but the difference between them is clear.

7. LINEAR APPROXIMATION

The stochastic Boltzmann equation given by Eq. (39) is the nonlinear generalization of the Boltzmann–Langevin equation[5] of Fox and Uhlenbeck[4] and Bixon and Zwanzig.[5] It reduces to their equation near the (absolute) equilibrium, as we now show.

If the conserved quantities of Eq. (44) are occasionally independent of the space coordinates, the associated currents of Eq. (46) vanish and the local equilibrium distribution that is constructed from these conserved quantities is just the equilibrium distribution. In this manner, the analog

of the Maxwell distribution function may be introduced into the theory of the stochastic Boltzmann equation.

The linear approximation

$$g = F_0[1 + h] \tag{66}$$

where F_0 is the Maxwell distribution, may be considered. Equation (39) gives

$$\frac{\partial}{\partial t} h + \mathbf{p} \frac{\partial}{\partial \mathbf{q}} h = n^2 I_0[h] + r_0(\mathbf{p}, \mathbf{q}, t) \tag{67}$$

where $\mathscr{I}_0$ is the linearized collision operator, the definition of which is given by Eq. (53) only if the $F(\mathbf{p}, \mathbf{C}, t)$'s are replaced by F_0's. The correlation function of $r_0(\mathbf{p}, \mathbf{q}, t)$ is given by Eq. (41), where the $g(\mathbf{p}, \mathbf{q}, t)$'s are to be replaced by F_0's. The random quantity $r_0(\mathbf{p}, \mathbf{q}, t)$ is a Gaussian process by the same reasoning as in the preceding section. It is characterized by

$$\begin{aligned} \overline{r_0(\mathbf{p}, \mathbf{q}, t) r_0(\mathbf{p}', \mathbf{q}', t')} &= \tfrac{1}{2} \delta(t - t') \, \delta(\mathbf{q} - \mathbf{q}') \int \cdots \int \Delta[\delta_{\mathbf{p}}] \, \Delta[\delta_{\mathbf{p}'}] \\ &\quad \times W_B(\mathbf{p}_1, \mathbf{p}_2, \mathbf{p}_1', \mathbf{p}_2') F_0(\mathbf{p}_1) F_0(\mathbf{p}_2) \\ &\quad \times d\mathbf{p}_1 \, d\mathbf{p}_2 \, d\mathbf{p}_1' \, d\mathbf{p}_2' \\ &= 2 \, \delta(t - t') \, \delta(\mathbf{q} - \mathbf{q}') \int d\mathbf{p}_1 \, \delta(\mathbf{p} - \mathbf{p}_1) n^2 I_0[\delta(\mathbf{p}' - \mathbf{p}_1)] \end{aligned} \tag{68}$$

Equations (67) and (68) give the linear approximation.

8. CONCLUSION AND DISCUSSION

On the assumption of the master equation, the stochastic Boltzmann equation for homogeneous fluctuations is derived rigorously in Eq. (31), the random quantity of which is a Gaussian process in the thermodynamic limit.

The stochastic Boltzmann equation for the general case is derived on the assumption of a kinetic equation that is similar to that of Hopfield and Bastin.[27] The random quantity is a Poissonian-like process in this case.

The linear approximation of this equation is just the Boltzmann–Langevin equation of Bixon and Zwanzig and of Fox and Uhlenbeck. The Langevin force is a Gaussian white noise.

Its approximation near the local equilibrium gives the equations of fluid dynamics of Landau and Lifshitz,[1] the random quantities of which are Gaussian processes.

Onuki arrived at almost the same results as Eqs. (39) and (41).[23] Perturbative solutions of Eq. (31) in which the random quantity is physically

small are expected to reproduce the results of van Kampen[9] and of Logan and Kac.[11]

A generalization of the stochastic Boltzmann equation to the hard-sphere system is given in Appendix A. Another generalization to mixed gases or to chemically reacting gases is given in a succeeding paper, in which hydrodynamic fluctuations in these gases are also discussed.

APPENDIX A. STOCHASTIC ENSKOG EQUATION

The derivation of the stochastic Boltzmann equation may be performed without the use of the local collision approximation. The kinetic equation of the hard-sphere system is given by the set of equations (6), (7), and (10).

In the spirit of the Kramers–Moyal expansion[31] the kinetic equation is rewritten as

$$\frac{\partial}{\partial t}f + \sum_{j=1}^{N} \mathbf{v}_j \frac{\partial}{\partial \mathbf{x}_j} f = \sum_{(i,j)} \sum_{\mathbf{n},\mathbf{m}} \left(\prod_{\lambda=x,y,z} \frac{(-1)^{n_\lambda + m_\lambda}}{n_\lambda!\, m_\lambda!} \frac{\partial^{n_\lambda}}{\partial x_{i\lambda}^{n_\lambda}} \frac{\partial^{m_\lambda}}{\partial v_{i\lambda}^{m_\lambda}} \right) \times \alpha_{\mathbf{nm}}(\mathbf{v}_i, \mathbf{v}_j, \mathbf{x}_i, \mathbf{x}_j) f \quad \text{(A1)}$$

where $\mathbf{n} = (n_x, n_y, n_z)$, $\mathbf{m} = (m_x, m_y, m_z)$, and the summations over $\mathbf{n}$ and $\mathbf{m}$ are over integer values of all the components. In Eq. (A1), $\alpha_{\mathbf{nm}}$ is the derivate moment defined by

$$\begin{aligned} \alpha_{\mathbf{nm}}(\mathbf{v}_i, \mathbf{v}_j, \mathbf{x}_i, \mathbf{x}_j) &= 0 && \text{if } \mathbf{n} = \mathbf{m} = \mathbf{0} \\ &= \iint \left\{ \prod_{\lambda} (x'_{i\lambda} - x_{i\lambda})^{n_\lambda} (v'_{i\lambda} - v_{i\lambda})^{m_\lambda} \right\} && \text{(A2)} \\ &\quad \times W(\mathbf{v}_i', \mathbf{v}_j', \mathbf{v}_i, \mathbf{v}_j, \mathbf{x}_i, \mathbf{x}_j)\, d\mathbf{v}_i'\, d\mathbf{v}_j' && \text{otherwise} \end{aligned}$$

If the distribution function at a time t has the special form

$$f(\mathbf{x}_1, \mathbf{v}_1, \mathbf{x}_2, \mathbf{v}_2, \ldots, \mathbf{x}_N, \mathbf{v}_N) = \sum_{j=1}^{N} \delta(\mathbf{x}_i - \mathbf{x}_{it})\, \delta(\mathbf{v}_i - \mathbf{v}_{it}) \quad \text{(A3)}$$

with parameters $\mathbf{x}_{it}$ and $\mathbf{v}_{it}$, the expectation value of the quantity $P = P(\mathbf{x}_1, \mathbf{v}_1, \ldots, \mathbf{x}_N, \mathbf{v}_N)$ at a slightly later time $t + \tau$ is given by

$$\begin{aligned} \langle P \rangle_{t+\tau} = P_t &+ \tau \sum_{i\mu} v_{it\mu} \frac{\partial}{\partial x_{it\mu}} P_t \\ &+ \tau \sum_{(i,j)} \sum_{\mathbf{n},\mathbf{m}} \alpha_{\mathbf{nm}}(\mathbf{v}_{it}, \mathbf{v}_{jt}, \mathbf{x}_{it}, \mathbf{x}_{jt}) \\ &\times \left(\prod_{\lambda} \frac{1}{n_\lambda!\, m_\lambda!} \frac{\partial^{n_\lambda}}{\partial x_{it\lambda}^{n_\lambda}} \frac{\partial^{m_\lambda}}{\partial v_{it\lambda}^{m_\lambda}} \right) P_t \end{aligned} \quad \text{(A4)}$$

where

$$P_t = P(\mathbf{x}_{1t}, \mathbf{v}_{1t}, \ldots, \mathbf{x}_{Nt}, \mathbf{v}_{Nt})$$

As special cases of Eq. (A4), this gives

$$\langle \mathbf{x}_i \rangle_{t+\tau} = \mathbf{x}_{it} + \tau \mathbf{v}_{it} \tag{A5}$$

$$\langle \mathbf{v}_i \rangle_{t+\tau} = \mathbf{v}_{it} + \tau \mathbf{F}_{it} \tag{A6}$$

where

$$F_{it\mu} = \sum_{j=1}^{N} \alpha_{\mathbf{e}_\mu 0}(\mathbf{v}_{it}, \mathbf{v}_{jt}, \mathbf{x}_{it}, \mathbf{x}_{jt}) \tag{A7}$$

with the use of unit vectors $\mathbf{e}_\mu$.

The Langevin-type equations of motion are given by

$$\mathbf{x}_{it+\tau} = \mathbf{x}_{it} + \tau \mathbf{v}_{it} \tag{A8}$$

$$\mathbf{v}_{it+\tau} = \mathbf{v}_{it} + \tau \mathbf{F}_{it} + \boldsymbol{\xi}_{it} \tag{A9}$$

the random quantities $\boldsymbol{\xi}_{it}$ of which are characterized by the equivalence relation

$$\tilde{P}_{t+\tau} = \langle P \rangle_{t+\tau} \tag{A10}$$

for arbitrary variable P.

If $P = \mathbf{v}_i$, Eq. (A10) gives

$$\overline{\boldsymbol{\xi}}_{it} = 0 \tag{A11}$$

and if $P = \prod_\lambda \mathbf{v}_{i\lambda}^{m_\lambda}$ or $P = \prod_\lambda v_{i\lambda}^{m_\lambda} v_{j\lambda}^{m_\lambda'}$ $(i \neq j)$, it gives

$$\overline{\prod_\lambda \xi_{i\lambda}^{m_\lambda}} = \tau \sum_{j=1}^{N} \alpha_{\mathbf{m}0}(\mathbf{v}_{it}, \mathbf{v}_{jt}, \mathbf{x}_{it}, \mathbf{x}_{jt}) \quad \text{if} \quad \sum_\lambda m_\lambda > 1 \tag{A12}$$

$$\overline{\prod_\lambda \xi_{i\lambda}^{m_\lambda} \xi_{j\lambda}^{m_\lambda'}} = \tau \alpha_{\mathbf{m}\mathbf{m}'}(\mathbf{v}_{it}, \mathbf{v}_{jt}, \mathbf{x}_{it}, \mathbf{x}_{jt})$$

$$\text{if} \quad i \neq j, \quad \sum_\lambda m_\lambda \geqslant 1, \quad \sum_\lambda m_\lambda' \geqslant 1 \tag{A13}$$

respectively. Successively, it also gives

$$\overline{\prod_\lambda \xi_{i\lambda}^{m_\lambda} \xi_{j\lambda}^{m_\lambda'} \xi_{k\lambda}^{m_\lambda''} \cdots} = 0 \quad \text{if} \quad i \neq j \neq k \neq i \tag{A14}$$

The simple result of Eq. (A14) reflects the fact that only the two-particle collisions are taken into account in the original kinetic equation. Even if a complicated expression containing spatial coordinates is chosen for P in Eq. (A10), it yields no other relations than Eqs. (A11)–(A14), because the additional τ dependence of $P_{t+\tau}$ due to that of the space coordinates is cancelled by the contribution of the drift term of Eq. (A4) to $\langle P \rangle_{t+\tau}$. The condition of Eq. (A10) is now realized by the assumptions of Eqs. (A11)–(A14).

It may be concluded that the set of Langevin-type equations [Eqs. (A8) and (A9)] is stochastically equivalent to Eq. (A1) if Eqs. (A11)–(A14) are assumed. We mean by the term "stochastically equivalent" that the equivalence relation of Eq. (A10) holds.

Since each particle moves according to Eqs. (A8) and (A9), a one-particle quantity

$$Q = \sum_{i=1}^{N} \sum_{\mathbf{nm}} Q(\mathbf{n}, \mathbf{m}) \prod_{\lambda} x_{i\lambda}^{n_\lambda} v_{i\lambda}^{m_\lambda} \tag{A15}$$

develops in time following the relation

$$Q_t = \sum_{i=1}^{N} \sum_{\mathbf{nm}} Q(\mathbf{n}, \mathbf{m}) \prod_{\lambda} x_{it\lambda}^{n_\lambda} v_{it\lambda}^{m_\lambda} \tag{A16}$$

Its change in a short interval τ is given by

$$Q_{t+\tau} - Q_t = \tau U_{Qt} + \tau V_{Qt} + W_{Qt} + O(\tau^2) \tag{A17}$$

where

$$U_{Qt} = \sum_{i} \sum_{\mathbf{nm}} Q(\mathbf{n}, \mathbf{m}) \sum_{\mu} v_{it\mu} \frac{\partial}{\partial x_{it\mu}} \prod_{\lambda} x_{it\lambda}^{n_\lambda} v_{it\lambda}^{m_\lambda} \tag{A18}$$

$$\begin{aligned} V_{Qt} = \sum_{i} \sum_{\mathbf{nm}} Q(\mathbf{n}, \mathbf{m}) \Bigg\{ & \sum_{\mu} F_{i\mu} \frac{\partial}{\partial v_{it\mu}} \prod_{\lambda} x_{it\lambda}^{n_\lambda} v_{it\lambda}^{m_\lambda} \\ & + \sum_{m'} \left[\prod_{\lambda} \binom{m_\lambda}{m_\lambda'} x_{it\lambda}^{n_\lambda} v_{it\lambda}^{m_\lambda - m_\lambda'} \right] \sum_{j} \alpha_{\mathbf{m}'\mathbf{0}}(\mathbf{v}_{it}, \mathbf{v}_{jt}, \mathbf{x}_{it}, \mathbf{x}_{jt}) \Bigg\} \end{aligned} \tag{A19}$$

$$W_{Qt} = \sum_{i} \sum_{\mathbf{nm}} Q(\mathbf{n}, \mathbf{m}) \{X - \bar{X}\}$$

$$X = \prod_{\lambda} x_{it\lambda}^{n_\lambda} (v_{it\lambda} + \xi_{it\lambda})^{m_\lambda} \tag{A20}$$

In Eq. (A19), the primed sum means summation with the condition $\sum m_\mu' > 1$.

With the aid of Eqs. (A2) and (A7), V_{Qt} is calculated as

$$\begin{aligned} V_{Qt} = \sum_{\mathbf{nm}} Q(\mathbf{n}, \mathbf{m}) \int \cdots \int & \left\{ \prod_{\lambda} q_\lambda^{n_\lambda} (p_\lambda')^{m_\lambda} - \prod_{\lambda} q_\lambda^{n_\lambda} p_\lambda^{m_\lambda} \right\} \\ & \times W(\mathbf{p}', \mathbf{p}_1', \mathbf{p}, \mathbf{p}_1, \mathbf{q}, \mathbf{q}_1) g(\mathbf{p}, \mathbf{q}, t) g(\mathbf{p}_1, \mathbf{q}_1, t) \\ & \times d\mathbf{p}\, d\mathbf{p}_1\, d\mathbf{p}'\, d\mathbf{p}_1'\, d\mathbf{q}\, d\mathbf{q}_1 \end{aligned} \tag{A19'}$$

where

$$g(\mathbf{p}, \mathbf{q}, t) = \sum_{i=1}^{N} \delta(\mathbf{q} - \mathbf{x}_{it})\, \delta(\mathbf{p} - \mathbf{v}_{it}) \tag{A21}$$

is the one-particle distribution.

By the use of the formal identities

$$Q_t = \sum_{\mathbf{nm}} Q(\mathbf{n}, \mathbf{m}) \int\int \left(\prod_\lambda q_\lambda^{n_\lambda} p_\lambda^{m_\lambda}\right) g(\mathbf{p}, \mathbf{q}, t)\, d\mathbf{p}\, d\mathbf{q}$$

$$W_{Qt} = \sum_{\mathbf{nm}} Q(\mathbf{n}, \mathbf{m}) \int\int \left(\prod_\lambda q_\lambda^{n_\lambda} p_\lambda^{m_\lambda}\right) R(\mathbf{p}, \mathbf{q}, t)\, d\mathbf{p}\, d\mathbf{q} \tag{A22}$$

where

$$R(\mathbf{p}, \mathbf{q}, t) = \sum_{i=1}^{N} \delta(\mathbf{q} - \mathbf{x}_{it})\{\delta(\mathbf{p} - \mathbf{v}_{it} - \xi_{it}) - \overline{\delta(\mathbf{p} - \mathbf{v}_{it} - \xi_{it})}\}$$

We can transform Eq. (A17) to the following:

$$\begin{aligned} g(\mathbf{p}, \mathbf{q}, t + \tau) = g(\mathbf{p}, \mathbf{q}, t) &- \tau \sum_\lambda p_\lambda \frac{\partial}{\partial q_\lambda} g(\mathbf{p}, \mathbf{q}, t) \\ &+ \tau \int\int\int\int \{W(\mathbf{p}, \mathbf{p}_1, \mathbf{p}', \mathbf{p}_1', \mathbf{q}, \mathbf{q}_1) g(\mathbf{p}', \mathbf{q}, t) g(\mathbf{p}_1', \mathbf{q}_1, t) \\ &- W(\mathbf{p}', \mathbf{p}_1', \mathbf{p}, \mathbf{p}_1, \mathbf{q}, \mathbf{q}_1) g(\mathbf{p}, \mathbf{q}, t) g(\mathbf{p}_1, \mathbf{q}_1, t)\} \\ &\times d\mathbf{p}_1\, d\mathbf{p}'\, d\mathbf{p}_1'\, d\mathbf{q}_1 + R(\mathbf{p}, \mathbf{q}, t) \end{aligned} \tag{A23}$$

In the limit of $\tau \to 0$, Eq. (A23) becomes

$$\frac{\partial}{\partial t} g + \mathbf{p} \frac{\partial}{\partial \mathbf{q}} g = J_H(g) + r(\mathbf{p}, \mathbf{q}, t) \tag{A24}$$

where

$$\begin{aligned} J_H(g) = \int\int\int\int &\{W(\mathbf{p}, \mathbf{p}_1, \mathbf{p}', \mathbf{p}_1'. \mathbf{q}, \mathbf{q}_1) g(\mathbf{p}', \mathbf{q}, t) g(\mathbf{p}_1', \mathbf{q}_1, t) \\ &- W(\mathbf{p}', \mathbf{p}_1', \mathbf{p}, \mathbf{p}_1, \mathbf{q}, \mathbf{q}_1) g(\mathbf{p}, \mathbf{q}, t) g(\mathbf{p}_1, \mathbf{q}_1, t)\} \\ &\times d\mathbf{p}_1\, d\mathbf{p}'\, d\mathbf{p}_1'\, d\mathbf{q}_1 \end{aligned} \tag{A25}$$

and $r(\mathbf{p}, \mathbf{q}, t)$ is defined as the limiting form of $R(\mathbf{p}, \mathbf{q}, t)$.

The correlation functions of $r(\mathbf{p}, \mathbf{q}, t)$ are calculated from Eqs. (A11)–(A14) as follows: First, we obtain

$$\overline{W_{Qt} W_{Q't}} = \sum_{\mathbf{m}, \mathbf{n}, \mathbf{n}', \mathbf{m}'} Q(\mathbf{n}, \mathbf{m}) Q'(\mathbf{n}', \mathbf{m}') \sum_{ij} X_{ij} + O(\tau^2)$$

$$X_{ij} = \overline{\prod_\lambda x_{it\lambda}^{n_\lambda} x_{jt\lambda}^{n_\lambda'} (v_{it\lambda} + \xi_{it\lambda})^{m_\lambda} (v_{jt\lambda} + \xi_{jt\lambda})^{m_\lambda'}} \tag{A26}$$

In Eq. (A26), terms X_{ii} and X_{ij} $(i \neq j)$ are to be calculated separately with the use of Eqs. (A12) and (A13) and are to be summed. We have

$$\begin{aligned}\sum_{ij} X_{ij} = \tau \sum_{ij} \iint \left\{ \prod_{\lambda} x_{it\lambda}^{n_\lambda + n_\lambda'} p_{1\lambda}^{m_\lambda + m_\lambda'} - \prod_{\lambda} x_{it\lambda}^{n_\lambda + n_\lambda'} v_{it\lambda}^{m_\lambda + m_\lambda'} \right\} \\ \times W(\mathbf{p}, \mathbf{p}_1, \mathbf{v}_{it}, \mathbf{v}_{jt}, \mathbf{x}_{it}, \mathbf{x}_{jt})\, d\mathbf{p}\, d\mathbf{p}_1 \\ + \tau \sum_{ij} \iint \left\{ \prod_{\lambda} x_{it\lambda}^{n_\lambda} x_{jt\lambda}^{n_\lambda'} p_{\lambda}^{m_\lambda} p_{1\lambda}^{m_\lambda'} \right. \\ \left. - \prod_{\lambda} x_{it\lambda}^{n_\lambda} x_{jt\lambda}^{n_\lambda'} v_{it\lambda}^{m_\lambda} v_{1\lambda}^{m_\lambda'} \right\} W(\mathbf{p}, \mathbf{p}_1, \mathbf{v}_{it}, \mathbf{v}_{jt}, \mathbf{x}_{it}, \mathbf{x}_{jt}) \\ = \tfrac{1}{2}\tau \int \cdots \int \left\{ \prod_{\lambda} X_{\lambda}^{n_\lambda} (X_\lambda')^{n'}{}_{\lambda} P_{\lambda}^{m_\lambda} (P_\lambda')^{m'}{}_{\lambda} \right\} \Delta[\delta_{\mathbf{XP}}]\, \Delta[\delta_{\mathbf{X'P'}}] \\ \times W(\mathbf{p}, \mathbf{p}_1, \mathbf{p}', \mathbf{p}_1', \mathbf{q}, \mathbf{q}_1) g(\mathbf{p}', \mathbf{q}, t) g(\mathbf{p}_1', \mathbf{q}_1, t) \\ \times d\mathbf{X}\, d\mathbf{X}'\, d\mathbf{P}\, d\mathbf{P}'\, d\mathbf{p}\, d\mathbf{p}_1\, d\mathbf{p}'\, d\mathbf{p}_1'\, d\mathbf{q}\, d\mathbf{q}_1 \qquad \text{(A27)}\end{aligned}$$

where

$$\begin{aligned}\Delta[\delta_{\mathbf{XP}}] = \delta(\mathbf{X} - \mathbf{q})\, \delta(\mathbf{P} - \mathbf{p}) + \delta(\mathbf{X} - \mathbf{q}_1)\, \delta(\mathbf{P} - \mathbf{p}_1) \\ - \delta(\mathbf{X} - \mathbf{q})\, \delta(\mathbf{P} - \mathbf{p}') - \delta(\mathbf{X} - \mathbf{q}_1)\, \delta(\mathbf{P} - \mathbf{p}_1') \qquad \text{(A28)}\end{aligned}$$

The result of Eqs. (A26) and (A27) is reduced in the limit $\tau \to 0$ to the expression

$$\begin{aligned}\overline{r(\mathbf{p}, \mathbf{q}, t) r(\mathbf{p}', \mathbf{q}', t')} \\ = \tfrac{1}{2}\delta(t - t') \int \cdots \int \Delta[\delta_{\mathbf{qp}}]\, \Delta[\delta_{\mathbf{q'p'}}] \\ \times W(\mathbf{p}, \mathbf{p}_1, \mathbf{p}', \mathbf{p}_1', \mathbf{q}, \mathbf{q}_1) g(\mathbf{p}', \mathbf{q}, t) g(\mathbf{p}_1', \mathbf{q}_1, t) \\ \times d\mathbf{p}\, d\mathbf{p}_1\, d\mathbf{p}'\, d\mathbf{p}_1'\, d\mathbf{q}\, d\mathbf{q}_1 \qquad \text{(A29)}\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}\overline{r(\mathbf{p}, \mathbf{q}, t) r(\mathbf{p}', \mathbf{q}', t') r(\mathbf{p}'', \mathbf{q}'', t'')} \\ = \tfrac{1}{2}\delta(t - t')\, \delta(t' - t'') \int \cdots \int \Delta[\delta_{\mathbf{qp}}]\, \Delta[\delta_{\mathbf{q'p'}}]\, \Delta[\delta_{\mathbf{q''p''}}] \\ \times W(\mathbf{p}, \mathbf{p}_1, \mathbf{p}', \mathbf{p}_1', \mathbf{q}, \mathbf{q}_1) g(\mathbf{p}', \mathbf{q}, t) g(\mathbf{p}_1', \mathbf{q}_1, t) \\ \times d\mathbf{p}\, d\mathbf{p}_1\, d\mathbf{p}'\, d\mathbf{p}_1'\, d\mathbf{q}\, d\mathbf{q}_1 \qquad \text{(A30)}\end{aligned}$$

Equation (A29) [Eq. (A30)] reduces to Eq. (41) [Eq. (42)] in the limit of vanishing molecular radius.

Equation (A24) is the generalization of the stochastic Boltzmann equation in the hard-sphere system. The generalization of the (usual) Boltzmann equation in that system is known as the Enskog equation

$$\frac{\partial}{\partial t}f + \mathbf{p}\frac{\partial}{\partial \mathbf{q}}f = J_E(f)$$

$$J_E(f) = \int d\mathbf{p}_2\, \sigma^2 \int d\hat{\boldsymbol{\sigma}}\, (\mathbf{p}_{21}\cdot\hat{\boldsymbol{\sigma}})$$
$$\{\chi(\sigma|n(\mathbf{q}_1 + \tfrac{1}{2}\sigma\hat{\boldsymbol{\sigma}}))F(\mathbf{p}_1, \mathbf{q}_1)F(\mathbf{p}_2, q_1 + \sigma\hat{\boldsymbol{\sigma}})$$
$$- \chi(\sigma|n(\mathbf{q}_1 - \tfrac{1}{2}\sigma\hat{\boldsymbol{\sigma}}))F(\mathbf{p}_1, \mathbf{q}_1)F(\mathbf{p}_2, \mathbf{q}_1 - \sigma\hat{\boldsymbol{\sigma}})\} \qquad \text{(A31)}$$

where χ is the equilibrium pair distribution function.[32] It is interesting that the deterministic part of Eq. (A24) differs from the Enskog equation by the χ factor.

The normal solutions of Eq. (A24), following the arguments of the hydrodynamic approximation of the stochastic Boltzmann equation, have the form of

$$g(\mathbf{p}, \mathbf{q}, t) = n(\mathbf{q})h(\mathbf{p}, \mathbf{q}, t) \qquad \text{(A32)}$$

where $n(\mathbf{q})$ is the microscopic density

$$n(\mathbf{q}) = \sum_{i=1}^{N} \delta(\mathbf{q} - \mathbf{x}_{it}) \qquad \text{(A33)}$$

instead of the usual one. Because of the hard-sphere repulsion, this density shows the strong correlation

$$n(\mathbf{q})n(\mathbf{q}') = 0 \qquad \text{if} \quad |\mathbf{q} - \mathbf{q}'| < \sigma \qquad \text{(A34)}$$

which the usual density does not. The one-particle description of the hard-sphere system seems possible only when this correlation is taken into account in some approximate way. When we imagine a system of particles with vanishing molecular radius and consider that this system represents approximately the system of hard spheres, we may suppose that the microscopic densities of the two systems are related by

$$n(\mathbf{q}) \simeq n_0(\mathbf{q}) \qquad \text{(A35)}$$

when $n(\mathbf{q})$ appears alone, but by

$$n(\mathbf{q})n(\mathbf{q}') \simeq \chi\left(n_0\left|\frac{\mathbf{q} + \mathbf{q}'}{2}\right.\right)n_0(\mathbf{q})n_0(\mathbf{q}') \qquad \text{(A36)}$$

when it appears pairwise. In Eq. (A35), $n_0(\mathbf{q})$ is the microscopic density of the reference system.

With the approximation of Eqs. (A35) and (A36), Eq. (A24) becomes the stochastic Enskog equation

$$\frac{\partial}{\partial t}\tilde{g} + \mathbf{p}\frac{\partial}{\partial \mathbf{q}}\tilde{g} = J_E(\tilde{g}) + \tilde{r}(\mathbf{p}, \mathbf{q}, t) \tag{A37}$$

where $\tilde{g}$ is the effective one-particle distribution

$$\tilde{g} = n_0(\mathbf{q})h(\mathbf{p}, \mathbf{q}, t) \tag{A38}$$

The random quantity $\tilde{r}(\mathbf{p}, \mathbf{q}, t)$ is characterized by Eqs. (A29) and (A30) only if the following substitution is made:

$$g(\mathbf{p}', \mathbf{q}, t)g(\mathbf{p}_1', \mathbf{q}_1, t) \simeq \chi\left(n_0\left|\frac{\mathbf{q} + \mathbf{q}'}{2}\right.\right)\tilde{g}(\mathbf{p}', \mathbf{q}, t)\tilde{g}(\mathbf{p}_1', \mathbf{q}_1, t) \tag{A39}$$

It is easy to see that Eq. (A37) reproduces the kinetic equation of the correlation function of Konijnendijk and van Leeuwen[33] in the linear approximation. I shall present a detailed analysis of Eq. (A37) in the near future.

APPENDIX B. DERIVATION OF EQS. (25) AND (26)

A definition of the δ-function is given by the identity

$$\int F(\boldsymbol{p})\,\delta(\boldsymbol{p} - \boldsymbol{v}_t)\,d\boldsymbol{p} = F(\boldsymbol{v}_t) \tag{B1}$$

for an arbitrary function $F(\cdot)$, which we assume to be analytic. Equation (16), when raised to the nth power, gives

$$\boldsymbol{v}_{t+\tau}^n = \boldsymbol{v}_t^n + n\tau\boldsymbol{v}_t^{n-1}\alpha_1(\boldsymbol{v}_t) + \sum_{k=1}\binom{n}{k}\boldsymbol{v}_t^{n-k}\xi^k(\boldsymbol{v}_t, \tau) \tag{B2}$$

if terms of order $O(\tau^2)$ are neglected. The right-hand side of Eq. (B2) is rewritten with the use of Eq. (19) as

$$\begin{aligned}\boldsymbol{v}_{t+\tau}^n &= \boldsymbol{v}_t^n + \tau\sum_{k=1}^{n}\binom{n}{k}\boldsymbol{v}_t^{n-k}\alpha_k(\boldsymbol{v}_t) + W_n(\boldsymbol{v}_t, \tau) \\ &= \boldsymbol{v}_t^n + \tau\int(\boldsymbol{p}^n - \boldsymbol{v}_t^n)w(\boldsymbol{p}, \boldsymbol{v}_t)\,d\boldsymbol{p} + W_n(\boldsymbol{v}_t, \tau)\end{aligned} \tag{B3}$$

where

$$W_n(\boldsymbol{v}_t, \tau) = \sum_{k=1}^{n}\binom{n}{k}\boldsymbol{v}_t^{n-k}\{\xi^k(\boldsymbol{v}_t, \tau) - \overline{\xi^k(\boldsymbol{v}_t, \tau)}\} \tag{B4}$$

Using Eq. (19) repeatedly, we have the following relations:

$$\overline{W_n(\boldsymbol{v}_t, \tau)} = 0 \tag{B5}$$

$$\overline{W_n(\boldsymbol{v}_t, \tau) W_m(\boldsymbol{v}_t, \tau)} = \tau \sum_{k=1}^{n} \sum_{l=1}^{m} \binom{n}{k} \binom{m}{l} \boldsymbol{v}_t^{n+m-k-l} \alpha_{k+l}(\boldsymbol{v}_t) + O(\tau^2) \tag{B6}$$

$$\overline{W_n(\boldsymbol{v}_t, \tau) W_m(\boldsymbol{v}_t, \tau) W_j(\boldsymbol{v}_t, \tau)} = \sum_{k=1}^{n} \sum_{l=1}^{m} \sum_{i=1}^{j} \binom{n}{k} \binom{m}{l} \binom{j}{i} \boldsymbol{v}_t^{n+m+j-k-l-i} \alpha_{k+l+i}(\boldsymbol{v}_t) + O(\tau^2) \tag{B7}$$

When Eq. (15) is substituted into the right-hand side of Eq. (B6) or Eq. (B7), it becomes

$$\begin{aligned} &\tau \iint d\boldsymbol{p}_1\, d\boldsymbol{p}_2 \sum_{k=1}^{n} \sum_{l=1}^{m} \binom{n}{k} \binom{m}{l} \boldsymbol{p}_2^{n+m-k-l} (\boldsymbol{p}_1 - \boldsymbol{p}_2)^{k+l} w(\boldsymbol{p}_1, \boldsymbol{p}_2)\, \delta(\boldsymbol{p}_2 - \boldsymbol{v}_t) \\ &\quad = \tau \iint d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, (\boldsymbol{p}_1{}^n - \boldsymbol{p}_2{}^n)(\boldsymbol{p}_1{}^m - \boldsymbol{p}_2{}^m) w(\boldsymbol{p}_1, \boldsymbol{p}_2)\, \delta(\boldsymbol{p}_2 - \boldsymbol{v}_t) \\ &\quad = \tau \int \cdots \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, \boldsymbol{x}_1{}^n\, \boldsymbol{x}_2{}^m \left\{ \prod_{i=1}^{2} [\delta(\boldsymbol{x}_i - \boldsymbol{p}_1) - \delta(\boldsymbol{x}_i - \boldsymbol{p}_2)] \right\} \\ &\qquad \times w(\boldsymbol{p}_1, \boldsymbol{p}_2)\, \delta(\boldsymbol{p}_2 - \boldsymbol{v}_t) \end{aligned} \tag{B6'}$$

or

$$\begin{aligned} &\tau \int \cdots \int d\boldsymbol{x}_1\, d\boldsymbol{x}_2\, d\boldsymbol{x}_3\, d\boldsymbol{p}_1\, d\boldsymbol{p}_2\, \boldsymbol{x}_1{}^n\, \boldsymbol{x}_2{}^m\, \boldsymbol{x}_3{}^j \\ &\qquad \times \left\{ \prod_{i=1}^{3} [\delta(\boldsymbol{x}_i - \boldsymbol{p}_1) - \delta(\boldsymbol{x}_i - \boldsymbol{p}_2)] \right\} w(\boldsymbol{p}_1, \boldsymbol{p}_2)\, \delta(\boldsymbol{p}_2 - \boldsymbol{v}_t) \end{aligned} \tag{B7'}$$

respectively.

Equations (B6') and (B7') show that there exists another random quantity $R(\boldsymbol{p}) = R(\boldsymbol{p}, \boldsymbol{v}_t, \tau)$, which satisfies

$$W_n(\boldsymbol{v}_t, \tau) = \int \boldsymbol{p}^n R(\boldsymbol{p})\, d\boldsymbol{p} \tag{B8}$$

Characterizations of $R(\boldsymbol{p})$ are given from Eqs. (B6') and (B7') as

$$\begin{aligned} \overline{R(\boldsymbol{p}_1) R(\boldsymbol{p}_2)} = \tau \iint d\boldsymbol{p}\, d\boldsymbol{p}' \left\{ \prod_{i=1}^{2} [\delta(\boldsymbol{p}_i - \boldsymbol{p}) - \delta(\boldsymbol{p}_i - \boldsymbol{p}')] \right\} \\ \times w(\boldsymbol{p}, \boldsymbol{p}')\, \delta(\boldsymbol{p}' - \boldsymbol{v}_t) \end{aligned} \tag{B9}$$

and

$$\begin{aligned} \overline{R(\boldsymbol{p}_1) R(\boldsymbol{p}_2) R(\boldsymbol{p}_3)} = \tau \iint d\boldsymbol{p}\, d\boldsymbol{p}' \left\{ \prod_{i=1}^{3} [\delta(\boldsymbol{p}_i - \boldsymbol{p}) - \delta(\boldsymbol{p}_i - \boldsymbol{p}')] \right\} \\ \times w(\boldsymbol{p}, \boldsymbol{p}')\, \delta(\boldsymbol{p}' - \boldsymbol{v}_t) \end{aligned} \tag{B10}$$

The general term may be guessed as

$$\overline{\prod_{i=1}^{n} R(p_i)} = \tau \iint dp\, dp' \left\{ \prod_{i=1}^{n} [\delta(p_i - p) - \delta(p_i - p')] \right\} \times w(p, p')\, \delta(p' - v_t)$$

Now, Eq. (B3) may be rewritten with the aid of the definition of the δ-function as

$$\delta(p - v_{t+\tau}) = \delta(p - v_t) + \tau \int \{w(p, p')\, \delta(p' - v_t) - w(p', p)\, \delta(p - v_t)\}\, dp' + R(p) \tag{B11}$$

In the limit of $\tau \to 0$, Eq. (B11) [Eq. (B9)] reduces to Eq. (25) [Eq. (26)]. The random quantity $r(p, t)$ is the limiting form of $R(p)$;

$$\int_t^{t+\tau} r(p, s)\, ds = R(p) \tag{B12}$$

The above proof is also valid for many-dimensional processes if the following replacements are made

$$n = \{n_{j\mu}\}, \qquad k = \{k_{j\mu}\}$$

$$v_t^n = \prod_{j\mu} v_{j\mu}^{n_{j\mu}}, \qquad \binom{n}{k} = \prod_{j\mu} \binom{n_{j\mu}}{k_{j\mu}}$$

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REFERENCES

1. L. D. Landau and E. M. Lifshitz, *Fluid Mechanics* (Pergamon Press, 1959), Chapter 17.
2. L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Pergamon Press, 1958), Chapter 12.
3. L. Onsager and S. Machlup, *Phys. Rev.* **91**:1505 (1953).
4. R. F. Fox and G. E. Uhlenbeck, *Phys. Fluids* **13**:1893, 2881 (1970).
5. M. Bixon and R. Zwanzig, *Phys. Rev.* **187**:267 (1969).
6. M. Kac, in *The Boltzmann Equation*, E. G. D. Cohen and W. Thirring, eds. (*Acta Physica Austriaca*, Suppl. X; Springer Verlag, 1973).
7. G. E. Uhlenbeck, in *The Boltzmann Equation*, E. G. D. Cohan and W. Thirring, eds. (*Acta Physica Austriaca*, Suppl. X; Springer Verlag, 1973).
8. R. F. Fox, *J. Math. Phys.* **19**:1993 (1978).
9. N. G. van Kampen, *Phys. Lett.* **50A**:237 (1974).

10. N. G. van Kampen, in *Fluctuation Phenomena in Solids*, R. E. Burgess, ed. (Academic Press, 1965).
11. J. Logan and M. Kac, *Phys. Rev. A* **13**:458 (1976).
12. R. Graham, *Phys. Rev. A* **10**:1762 (1964).
13. J. E. Keizer, *Phys. Fluids* **21**:198 (1978).
14. F. L. Hinton, *Phys. Fluids* **13**:857 (1970).
15. S. Watanabe, *Stochastic Differential Equations* (Sangyotosho Press, 1978) (in Japanese).
16. L. Arnold, *Stochastic Differential Equations* (Wiley, 1974).
17. G. E. Uhlenbeck, Appendix to M. Kac, *Probability and Related Topics in Physical Sciences* (Interscience, 1961).
18. M. Kac, *Probability and Related Topics in Physical Sciences* (Interscience, 1959).
19. R. Brout, *Physica* **22**:509 (1956).
20. A. Onuki, *J. Stat. Phys.* **19**:325 (1978).
21. A. V. Skorokhod, *Studies in the Theory of Random Processes* (Addison-Wesley, 1965), p. 34.
22. H. Ueyama, *Physica* **80A**:98 (1975).
23. A. Onuki, *J. Stat. Phys.* **18**:475 (1978).
24. R. Zwanzig, in *Lectures in Theoretical Physics*, W. E. Brittin *et al.*, eds. (Interscience, 1961), Vol. 3.
25. S. Chapman and T. G. Cowling, *The Mathematical Theory of Nonuniform Gases* (Cambridge Press, 1970).
26. L. Waldmann, in *Handbuch der Physik, Vol. 12* (Springer, Berlin, 1958).
27. J. J. Hopfield and A. J. F. Bastin, *Phys. Rev.* **168**:193 (1968).
28. M. H. Ernst, J. R. Dorfman, W. R. Hoegy, and J. M. J. van Leeuwen, *Physica* **45**:127 (1969).
29. P. Langevin, *Compt. Rend.* **1908**:530.
30. K. Itô, *Mem. Am. Math. Soc.* **4**:1 (1951).
31. J. E. Moyal, *J. Roy. Stat. Soc. B* **11**:150 (1949).
32. H. van Beijeren and M. H. Ernst, *Physica* **68**:437 (1973).
33. H. H. U. Konijnendijk and J. M. J. van Leeuwen, *Physica* **64**:342 (1973).